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1997 J. Phys. A: Math. Gen. 30 4161

(http://iopscience.iop.org/0305-4470/30/12/008)

J. Phys. A: Math. Gen. **30** (1997) 4161–4169. Printed in the UK PII: S0305-4470(97)80621-X

Lax pair formulation for one-dimensional Hubbard open chain with chemical potential

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Received 2 January 1997, in final form 26 March 1997

Abstract. The Lax pair for the one-dimensional Hubbard spin chain with open boundary conditions (BC) is explicitly constructed and two different kinds of boundary K-matrices compatible with the integrability of the model are obtained. Our construction provides an alternative and direct demonstration for the quantum integrability of the model and it is found that the model is related to a class of commuting transfer matrices of an equivalent coupled asymmetric six-vertex model with open BC. Our results show that the chemical potential term is indeed nontrivial for the underlying algebraic structure of the model.

1. Introduction

In the last decade, much attention has been spent studying strongly correlated electronic systems [1–8]. It has been found that those models, such as the one-dimensional (1D) Hubbard model, the supersymmetric t-J model and 1D Bariev model, etc exhibit different physical behaviour and possess different algebraic structures [2, 5, 9, 10] underlying the integrability.

On the other hand, Sklyanin [11] showed that there is a variant of the usual formalism of the quantum inverse scattering method (QISM) [12, 13] which can be used to describe systems on the finite interval with independent boundary conditions (BC) on each end. Central to his approach is the introduction of a new algebraic structure called the reflection equations (RE), which guarantee the integrability of systems with open BC. Although Sklyanin's argument was carried out only for the P and T invariant R-matrices, and Mezincescu and Nepomechi [14] extended Sklyanin formulism to the case of PT-invariant R-matrices, it is now known that the formulism may be extended to apply to any chains integrable by the quantum R-matrix approach [15].

As is well known, the traditional basis for applying the QISM to a completely integrable system is to represent the equations of motion of the system into Lax form. Following Izergin and Korepin [16], one may show that for a system with periodic BC, the existence of the quantum R-matrix allows one to express the original equations of motion in the Lax form. Meanwhile the explicit forms of Lax pairs for some physically important models have been given by many literatures [17–19]. A natural problem is that there must exist a variant of the usual Lax formulation to describe completely integrable quantum lattice spin chains with open BC [20–22]. The aim of this paper is to present an explicit construction

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of the Lax pair for the Hubbard model with a chemical potential term [23–25]. As a further result, we obtain its two kinds of boundary K-matrices compatible with the integrability of the model. Those seem to contribute the different boundary terms in the Hamiltonian of the model. We found that the model is related to a class of commuting transfer matrices of an equivalent coupled asymmetric six-vertex model with open BC.

The outline of the present paper is as follows. In section 2, we recall the Lax pair formulation of the QISM for spin chains with open BC. In section 3, the 1D Hubbard model with chemical potential and its equivalent spin chain with open BC are introduced. In section 4, the Lax pair for the open chain are explicitly constructed and corresponding boundary K-matrices are obtained. Finally, section 5 is devoted to the conclusion.

2. Lax pair formulation

Let us first recall the Lax pair formulation of the QISM for completely integrable lattice spin open chains. We consider an operator version of an auxiliary linear problem:

$$\begin{aligned}
&\Phi_{j+1} = L_j(u)\Phi_j \qquad j = 1, 2, \ldots, N \\
&\frac{\mathrm{d}}{\mathrm{d}t}\Phi_j = M_j(u)\Phi_j \qquad j = 2, 3, \ldots, N \\
&\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{N+1} = M_{N+1}(u)\Phi_{N+1} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Phi_1 = M_1(u)\Phi_1.
\end{aligned} \tag{1}$$

Where $L_j(u)$, $M_j(u)$, M_{N+1} and $M_1(u)$ are the matrices depending on the spectral parameter u, which does not depend on time t, and dynamical variables. Evidently, the consistency conditions for equation (1) yield the following Lax equations:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}L_j(u) = M_{j+1}(u)L_j(u) - L_j(u)M_j(u) \qquad j = 2, 3, \ldots, N-1 \\
&\frac{\mathrm{d}}{\mathrm{d}t}L_N(u) = M_{N+1}(u)L_N(u) - L_N(u)M_N(u) \\
&\frac{\mathrm{d}}{\mathrm{d}t}L_1(u) = M_2(u)L_1(u) - L_1(u)M_1(u).
\end{aligned} \tag{2}$$

If the equations of motion of the system can be expressed in the form of equation (2), provided the boundary K-matrices exist as the solutions of equations (5) and (6) below, then we insist that the lattice spin chain with open BC is completely integrable. In fact, it is readily shown from equation (2) that the transfer matrix

$$\tau(u) = \mathrm{Tr}(K_+(u)T(u)K_-(u)T^{-1}(-u)) \tag{3}$$

does not depend on time. Where $T(u)$ is the monodromy matrix

$$T(u) = L_N(u)\ldots L_1(u). \tag{4}$$

This only requires that the boundary K-matrices satisfy the following constraint conditions:

$$K_-(u)M_1(-u) = M_1(u)K_-(u) \tag{5}$$

and

$$\mathrm{Tr}(K_+(u)M_{N+1}(u)A_N(u)) = \mathrm{Tr}(K_+(u)A_N(u)M_{N+1}(-u)) \tag{6}$$

where

$$A_N(u) = L_N(u)\cdots L_1(u)K_-(u)L_1^{-1}(-u)\ldots L_N^{-1}(-u).$$

This implies that the system possesses an infinite number of conserved quantities.

3. The model

Let us consider the 1D Hubbard model with open BC determined by the Hamiltonian

$$H = -\sum_{j=1}^{N-1}\sum_{s}(a_{j+1s}^{+}a_{js} + a_{js}^{+}a_{j+1s}) - U\sum_{j=1}^{N}(n_{j\uparrow} - \tfrac{1}{2})(n_{j\downarrow} - \tfrac{1}{2}) + \mu\sum_{j=1}^{N}\sum_{s}(n_{js} - \tfrac{1}{2})$$
$$+p_{+}(2n_{1\uparrow} - 1) + p_{-}(2n_{1\downarrow} - 1) + q_{+}(2n_{N\uparrow} - 1) + q_{-}(2n_{N\downarrow} - 1). \qquad (7)$$

Here $p_{\pm}$ and $q_{\pm}$ are the constants describing the boundary effect. The boundary terms in the Hamiltonian equation (7) are specially chosen, and only related to the diagonal boundary K-matrices. U is the coupling constant describing Coulomb interaction and μ is the chemical potential. a_{js}^{+} and a_{js} are creation and annihilation operators with spins ($s =\uparrow$ or $\downarrow$) at site j and $n_{js} = a_{js}^{+}a_{js}$ is the density operator. They satisfy the anticommutation relations

$$\begin{aligned}&\{a_{js}, a_{j's'}\} = \{a_{js}^{+}, a_{j's'}^{+}\} = 0\\ &\{a_{js}, a_{j's'}^{+}\} = \delta_{jj'}\delta_{ss'}.\end{aligned} \qquad (8)$$

Applying the Jordan–Wigner transformation which relates fermion operators and spin operators,

$$a_{j\uparrow} = \exp\left(\mathrm{i}\pi\sum_{l=1}^{j-1}\sigma_l^{+}\sigma_l^{-}\right)\sigma_j^{-}$$

$$a_{j\uparrow}^{+} = \exp\left(\mathrm{i}\pi\sum_{l=1}^{j-1}\sigma_l^{+}\sigma_l^{-}\right)\sigma_j^{+}$$

$$a_{j\downarrow} = \exp\left(\mathrm{i}\pi\sum_{l=1}^{N}\sigma_l^{+}\sigma_l^{-}\right)\exp\left(\mathrm{i}\pi\sum_{l=1}^{j-1}\tau_l^{+}\tau_l^{-}\right)\tau_j^{-}$$

$$a_{j\downarrow}^{+} = \exp\left(\mathrm{i}\pi\sum_{l=1}^{N}\sigma_l^{+}\sigma_l^{-}\right)\exp\left(\mathrm{i}\pi\sum_{l=1}^{j-1}\tau_l^{+}\tau_l^{-}\right)\tau_j^{+}$$

where σ and τ are two species of Pauli matrices, they commute each other, we obtain the Hamiltonian of a spin model which is equivalent to the Hubbard model (7):

$$H = -\sum_{j=1}^{N-1}(\sigma_{j+1}^{+}\sigma_j^{-} + \mathrm{hc}) + (\sigma \to \tau) - \frac{U}{4}\sum_{j=1}^{N}\sigma_j^{z}\tau_j^{z} + \frac{\mu}{2}\sum_{j=1}^{N}(\sigma_j^{z} + \tau_j^{z})$$
$$+p_{+}\sigma_1^{z} + p_{-}\tau_1^{z} + q_{+}\sigma_N^{z} + q_{-}\tau_N^{z}. \qquad (9)$$

In the case $U = 0$ and $\mu = 0$, this Hamiltonian reduces to a pair of uncoupled XY open chains. Thus, the original problem reduces to the study of two identical Heisenberg XY spin chains with open BC coupled each other.

For our purpose, let us now recall some basic results for the model with periodic BC. In [25] it was shown that the two-dimensional (2D) covering model, consisting of two couple asymmetric six-vertex models, provides a one-parameter family of transfer matrices commuting with the Hamiltonian of the 1D Hubbard spin chain with periodic BC. The transfer matrix is the trace of the monodromy matrix $T(u)$ equation (4), i.e.

$$\tau(u) = \mathrm{Tr}_0\, L_{N0}(u)L_{N-10}(u)\ldots L_{10}(u). \qquad (10)$$

where

$$L_{j0}(u) = I_0 L_j^{(\sigma)}(u) \otimes L_j^{(\tau)}(u) I_0 \qquad j = 1, \ldots, N \qquad (11)$$

with

$$I_0 = \cosh\frac{h}{2} + \sigma_0^z \tau_0^z \sinh\frac{h}{2}$$

and

$$L_j^{(\sigma)}(u) = \begin{pmatrix} \frac{1}{2}(a_+ + b_+) + \frac{1}{2}(a_+ - b_+)\sigma_j^z & c\sigma_j^- \\ c\sigma_j^+ & \frac{1}{2}(a_- + b_-) - \frac{1}{2}(a_- - b_-)\sigma_j^z \end{pmatrix} \tag{12}$$

with

$$a_+ : b_+ : a_- : b_- : c = \xi\cos u : \xi^{-1}\sin u : \xi^{-1}\cos u : \xi\sin u : 1$$
$$\xi = \frac{\cos(u+\beta)}{\cos\beta\cos u} \qquad \mu = 2\tan\beta$$
$$U = \frac{2c^2}{a_+ b_+}\sinh 2h = \frac{2c^2}{a_- b_-}\sinh 2h.$$

Note that $L_j^{(\tau)}(u)$ has the same form as $L_j^{(\sigma)}(u)$ with τ's replacing σ's. The parameter h controls the strength of the interlayer interactions. The local monodromy matrix $L_j(u)$ acts in the tensor product of the physical space W_j and the auxiliary space V_0 $(= C^2 \otimes C^2)$, and the trace as well as the matrix products are carried out in the auxiliary space V_0.

A completely integrable model exhibits an infinite number of conserved currents commuting with each other. The explicit expression for the conserved currents may be obtained by an expansion of the transfer matrix in the power of u. As was shown by Zhou *et al* [19], the equations of motion derived from the Hamiltonian of the 1D Hubbard spin chain with periodic BC may be cast into the Lax formulation. For our convenience, we may show that L and M take the form:

$$L_j(u) = \begin{pmatrix} e^h\Theta_+(j)\Phi_+(j) & \Theta_+(j)\tau_j^- & \sigma_j^-\Phi_+(j) & e^h\sigma_j^-\tau_j^- \\ \Theta_+(j)\tau_j^+ & e^{-h}\Theta_+(j)\Phi_-(j) & e^{-h}\sigma_j^-\tau_j^+ & \sigma_j^-\Phi_-(j) \\ \sigma_j^+\Phi_+(j) & e^{-h}\sigma_j^+\tau_j^- & e^{-h}\Theta_-(j)\Phi_+ j & \Theta_-(j)\tau_j^- \\ e^h\sigma_j^+\tau_j^+ & \sigma_j^+\Phi_-(j) & \Theta_-(j)\tau_j^+ & e^h\Theta_-(j)\Phi_-(j) \end{pmatrix} \tag{13}$$

where

$$\Theta_+(j) = W_1 + W_2\sigma_j^z \qquad \Phi_+(j) = W_1 + W_2\tau_j^z$$
$$\Theta_-(j) = W_3 - W_4\sigma_j^z \qquad \Phi_-(j) = W_3 - W_4\tau_j^z$$

with

$$W_1 + W_2 = \xi\cos u \qquad W_1 - W_2 = \xi^{-1}\sin u$$
$$W_3 + W_4 = \xi^{-1}\cos u \qquad W_3 - W_4 = \xi\sin u$$

and

$$M_j(u) = \begin{pmatrix} B_j(1,1) & \alpha E(\tau_j^-, \tau_{j-1}^-) & \alpha E(\sigma_j^-, \sigma_{j-1}^-) & 0 \\ \alpha E(\tau_{j-1}^+, \tau_j^+) & B_j(2,2) & 0 & \alpha F(\sigma_{j-1}^-, \sigma_j^-) \\ \alpha E(\sigma_{j-1}^+, \sigma_j^+) & 0 & B_j(3,3) & \alpha F(\tau_{j-1}^-, \tau_j^-) \\ 0 & \alpha F(\sigma_j^+, \sigma_{j-1}^+) & \alpha F(\tau_j^+, \tau_{j-1}^+) & B_j(4,4) \end{pmatrix} \tag{14}$$

where

$$B_j(1,1) = \nu - i\mu + i\rho_+(\sigma_j^+\sigma_{j-1}^- + \tau_j^+\tau_{j-1}^-) + i\kappa_-(\sigma_j^-\sigma_{j-1}^+ + \tau_j^-\tau_{j-1}^+)$$
$$B_j(2,2) = -\nu + i\rho_+\sigma_j^+\sigma_{j-1}^- + i\rho_-\tau_j^-\tau_{j-1}^+ + i\kappa_-\sigma_j^-\sigma_{j-1}^+ + i\kappa_+\tau_j^+\tau_{j-1}^-$$
$$B_j(3,3) = -\nu + i\rho_-\sigma_j^-\sigma_{j-1}^+ + i\rho_+\tau_j^+\tau_{j-1}^- + i\kappa_+\sigma_j^+\sigma_{j-1}^- + i\kappa_-\tau_j^-\tau_{j-1}^+$$
$$B_j(4,4) = \nu + i\mu + i\rho_-(\sigma_j^-\sigma_{j-1}^+ + \tau_j^-\tau_{j-1}^+) + i\kappa_+(\sigma_j^+\sigma_{j-1}^- + \tau_j^+\tau_{j-1}^-)$$
$$E(\boldsymbol{x}, \boldsymbol{y}) = (e^h\xi\boldsymbol{x} + e^{-h}\xi^{-1}\boldsymbol{y}),\; F(\boldsymbol{x}, \boldsymbol{y}) = (e^h\xi^{-1}\boldsymbol{x} + e^{-h}\xi\boldsymbol{y})$$

with

$$\rho_\pm = -(1 - \xi^{\mp 2}\tan u) \qquad \kappa_\pm = -(1 + \xi^{\mp 2}\tan u)$$
$$\nu = \frac{\mathrm{i}}{4}U(2 - \cos 2u) \qquad \alpha = \frac{\mathrm{i}}{\cos u} \qquad \boldsymbol{x}, \boldsymbol{y} = \sigma_j^\pm, \sigma_{j-1}^\pm, \tau_j^\pm, \tau_{j-1}^\pm.$$

4. Lax pair and boundary K-matrix

In order to construct the matrices $M_1(u)$ and $M_{N+1}(u)$, one needs the following equations of motion

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\sigma_1^+ &= \mathrm{i}\left(\sigma_2^+\sigma_1^z - \frac{U}{2}\sigma_1^+\tau_1^z + 2p_+\sigma_1^+ + \mu\sigma_1^+\right)\\
\frac{\mathrm{d}}{\mathrm{d}t}\sigma_N^+ &= \mathrm{i}\left(\sigma_N^z\sigma_{N-1}^+ - \frac{U}{2}\sigma_N^+\tau_N^z + 2q_+\sigma_N^+ + \mu\sigma_N^+\right)\\
\frac{\mathrm{d}}{\mathrm{d}t}\sigma_1^- &= -\mathrm{i}\left(\sigma_2^-\sigma_1^z - \frac{U}{2}\sigma_1^-\tau_1^z + 2p_+\sigma_1^- + \mu\sigma_1^-\right)\\
\frac{\mathrm{d}}{\mathrm{d}t}\sigma_N^- &= -\mathrm{i}\left(\sigma_N^z\sigma_{N-1}^- - \frac{U}{2}\sigma_N^-\tau_N^z + 2q_+\sigma_N^- + \mu\sigma_N^-\right)\\
\frac{\mathrm{d}}{\mathrm{d}t}\sigma_1^z &= 2\mathrm{i}(\sigma_1^+\sigma_2^- - \sigma_2^+\sigma_1^-) \qquad \frac{\mathrm{d}}{\mathrm{d}t}\sigma_N^z = 2\mathrm{i}(\sigma_N^+\sigma_{N-1}^- - \sigma_N^-\sigma_{N-1}^+)
\end{aligned} \tag{15}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau = \frac{\mathrm{d}}{\mathrm{d}t}\sigma(\sigma \to \tau, p_+ \to p_-, q_+ \to q_-) \tag{16}$$

yielded by the Hamiltonian (9).

Noting that the bulk part of (9) coincides with the counterpart of the periodic chain. From equation (2), we can explicitly construct

$$M_1(u) = \begin{pmatrix} D_1(1,1) & M_-\tau_1^- & M_+\sigma_1^- & 0\\ N_-\tau_1^+ & D_1(2,2) & 0 & A_+\sigma_1^-\\ N_+\sigma_1^+ & 0 & D_1(3,3) & A_-\tau_1^-\\ 0 & B_+\sigma_1^+ & B_-\tau_1^+ & D_1(4,4) \end{pmatrix} \tag{17}$$

with

$$\begin{aligned}
D_1(1,1) &= -\frac{2\mathrm{i}}{\cos^2 u}(p_+\sigma_1^-\sigma_1^+ + p_-\tau_1^-\tau_1^+) - \mathrm{i}\mu + \nu\\
D_1(4,4) &= \frac{2\mathrm{i}}{\cos^2 u}(p_+\sigma_1^+\sigma_1^- + p_-\tau_1^+\tau_1^-) + \mathrm{i}\mu + \nu\\
D_1(2,2) &= -\frac{2\mathrm{i}}{\cos^2 u}(p_+\sigma_1^-\sigma_1^+ - p_-\tau_1^+\tau_1^-) - \nu\\
D_1(3,3) &= \frac{2\mathrm{i}}{\cos^2 u}(p_+\sigma_1^+\sigma_1^- - p_-\tau_1^-\tau_1^+) - \nu\\
M_\pm &= \frac{\mathrm{i}\xi}{\cos^2 u}(\mathrm{e}^h\cos u - \mathrm{e}^{-h}2p_\pm\sin u) \qquad N_\pm = \frac{\mathrm{i}\xi^{-1}}{\cos^2 u}(\mathrm{e}^{-h}\cos u + \mathrm{e}^h 2p_\pm\sin u)\\
A_\pm &= \frac{\mathrm{i}\xi}{\cos^2 u}(\mathrm{e}^{-h}\cos u - \mathrm{e}^h 2p_\pm\sin u) \qquad B_\pm = \frac{\mathrm{i}\xi^{-1}}{\cos^2 u}(\mathrm{e}^h\cos u - \mathrm{e}^{-h}2p_\pm\sin u)
\end{aligned}$$

and

$$M_{N+1}(u) = \begin{pmatrix} D_N(1,1) & N'_-\tau_N^- & N'_+\sigma_N^- & 0 \\ M'_-\tau_N^+ & D_N(2,2) & 0 & B'_+\sigma_N^- \\ M'_+\sigma_N^+ & 0 & D_N(3,3) & B'_-\tau_N^- \\ 0 & A'_+\sigma_N^+ & A'_-\tau_N^+ & D_N(4,4) \end{pmatrix} \tag{18}$$

with

$$D_N(1,1) = -\frac{2\mathrm{i}}{\cos^2 u}(q_+\sigma_1^-\sigma_1^+ + q_-\tau_1^-\tau_1^+) - \mathrm{i}\mu + \nu$$
$$D_N(4,4) = \frac{2\mathrm{i}}{\cos^2 u}(q_+\sigma_1^+\sigma_1^- + q_-\tau_1^+\tau_1^-) + \mathrm{i}\mu + \nu$$
$$D_N(2,2) = -\frac{2\mathrm{i}}{\cos^2 u}(q_+\sigma_1^-\sigma_1^+ - q_-\tau_1^+\tau_1^-) - \nu$$
$$D_N(3,3) = \frac{2\mathrm{i}}{\cos^2 u}(q_+\sigma_1^+\sigma_1^- - q_-\tau_1^-\tau_1^+) - \nu$$
$$M'_\pm = \frac{\mathrm{i}\xi}{\cos^2 u}(\mathrm{e}^h\cos u - \mathrm{e}^{-h}2q_\pm\sin u) \qquad N'_\pm = \frac{\mathrm{i}\xi^{-1}}{\cos^2 u}(\mathrm{e}^{-h}\cos u + \mathrm{e}^h 2q_\pm \sin u)$$
$$A'_\pm = \frac{\mathrm{i}\xi}{\cos^2 u}(\mathrm{e}^{-h}\cos u - \mathrm{e}^h 2q_\pm\sin u) \qquad B'_\pm = \frac{\mathrm{i}\xi^{-1}}{\cos^2 u}(\mathrm{e}^h\cos u + \mathrm{e}^{-h}2q_\pm\sin u).$$

Thus, we have obtained the Lax pair for 1D Hubbard spin open chain. This also gives a straightforward proof of the integrability of the model.

We now proceed to study the constraint conditions equations (5) and (6) to find the boundary K-matrix $K_\pm$. Let

$$K_\pm(u) = \begin{pmatrix} K_\pm(1,1) & 0 & 0 & 0 \\ 0 & K_\pm(2,2) & 0 & 0 \\ 0 & 0 & K_\pm(3,3) & 0 \\ 0 & 0 & 0 & K_\pm(4,4) \end{pmatrix}. \tag{19}$$

Substituting (17) and (19) into equation (5), one obtains 16 equations. By tedious algebraic calculation we obtain only two different kinds of solutions as follows.

Case I: If $p_+ = p_- = \zeta_-$

$$\begin{aligned} K_-(2,2) &= \frac{\xi'(\mathrm{e}^{-h}\cos u + \mathrm{e}^h 2\zeta_-\sin u)}{\xi(\mathrm{e}^h\cos u - \mathrm{e}^{-h}2\zeta_-\sin u)}K_-(1,1) \\ K_-(3,3) &= \frac{\xi'(\mathrm{e}^{-h}\cos u + \mathrm{e}^h 2\zeta_-\sin u)}{\xi(\mathrm{e}^h\cos u - \mathrm{e}^{-h}2\zeta_-\sin u)}K_-(1,1) \\ K_-(4,4) &= \frac{\xi'(\mathrm{e}^{h}\cos u + \mathrm{e}^{-h} 2\zeta_-\sin u)}{\xi(\mathrm{e}^{-h}\cos u - \mathrm{e}^{h}2\zeta_-\sin u)}K_-(2,2). \end{aligned} \tag{20}$$

Case II: If $p_+ = -p_- = \zeta_-$

$$\begin{aligned} K_-(2,2) &= \frac{\xi'(\mathrm{e}^{-h}\cos u - \mathrm{e}^h 2\zeta_-\sin u)}{\xi(\mathrm{e}^h\cos u + \mathrm{e}^{-h}2\zeta_-\sin u)}K_-(1,1) \\ K_-(3,3) &= \frac{\xi'(\mathrm{e}^{-h}\cos u + \mathrm{e}^h 2\zeta_-\sin u)}{\xi(\mathrm{e}^h\cos u - \mathrm{e}^{-h}2\zeta_-\sin u)}K_-(1,1) \\ K_-(4,4) &= \frac{\xi'(\mathrm{e}^{h}\cos u + \mathrm{e}^{-h} 2\zeta_-\sin u)}{\xi(\mathrm{e}^{-h}\cos u - \mathrm{e}^{h}2\zeta_-\sin u)}K_-(2,2). \end{aligned} \tag{21}$$

Where $\xi'(u) = \xi(-u)$. From those equations $K_-(u)$ shall be given in the following expressions.

Case I

$$
\begin{aligned}
K_-(1,1) &= \lambda_- \cos^2(u+\beta)(\mathrm{e}^{-h}\cos u - \mathrm{e}^{h}2\zeta_-\sin u)(\mathrm{e}^{h}\cos u - \mathrm{e}^{-h}2\zeta_-\sin u)\\
K_-(4,4) &= \lambda_- \cos^2(u-\beta)(\mathrm{e}^{h}\cos u + \mathrm{e}^{-h}2\zeta_-\sin u)(\mathrm{e}^{-h}\cos u + \mathrm{e}^{h}2\zeta_-\sin u)\\
K_-(2,2) &= \lambda_- \cos(u-\beta)\cos(u+\beta)(\mathrm{e}^{-h}\cos u + \mathrm{e}^{h}2\zeta_-\sin u)(\mathrm{e}^{-h}\cos u - \mathrm{e}^{h}2\zeta_-\sin u)\\
K_-(3,3) &= \lambda_- \cos(u-\beta)\cos(u+\beta)(\mathrm{e}^{-h}\cos u + \mathrm{e}^{h}2\zeta_-\sin u)(\mathrm{e}^{-h}\cos u - \mathrm{e}^{h}2\zeta_-\sin u).
\end{aligned}
\tag{22}
$$

Case II

$$
\begin{aligned}
K_-(1,1) &= \lambda_- \cos^2(u+\beta)(\mathrm{e}^{h}\cos u + \mathrm{e}^{-h}2\zeta_-\sin u)(\mathrm{e}^{h}\cos u - \mathrm{e}^{-h}2\zeta_-\sin u)\\
K_-(4,4) &= \lambda_- \cos^2(u-\beta)(\mathrm{e}^{h}\cos u + \mathrm{e}^{-h}2\zeta_-\sin u)(\mathrm{e}^{h}\cos u - \mathrm{e}^{-h}2\zeta_-\sin u)\\
K_-(2,2) &= \lambda_- \cos(u-\beta)\cos(u+\beta)(\mathrm{e}^{-h}\cos u - \mathrm{e}^{h}2\zeta_-\sin u)(\mathrm{e}^{h}\cos u - \mathrm{e}^{-h}2\zeta_-\sin u)\\
K_-(3,3) &= \lambda_- \cos(u-\beta)\cos(u+\beta)(\mathrm{e}^{-h}\cos u + \mathrm{e}^{h}2\zeta_-\sin u)(\mathrm{e}^{h}\cos u + \mathrm{e}^{-h}2\zeta_-\sin u).
\end{aligned}
\tag{23}
$$

Here ζ_- and λ_- are arbitrary constants describing the boundary effect.

In order to determine $K_+(u)$, one notes that

$$
A_N(u) = L_N(u)A_{N-1}(u)L_N^{-1}(-u) \tag{24}
$$

obviously, the matrix elements of A_{N-1} are independent of each other and commute with matrix elements of $L_N(u)$. Therefore one can assume that the corresponding coefficients of the matrix elements of $A_{N-1}(u)$ are equal on both sides of equation (6). It follows that:

Case I: If $q_+ = q_- = \zeta_+$

$$
\begin{aligned}
K_+(2,2) &= \frac{\xi(\mathrm{e}^{-h}\sin u - \mathrm{e}^{h}2\zeta_+\cos u)}{\xi'(\mathrm{e}^{h}\sin u + \mathrm{e}^{-h}2\zeta_+\cos u)}K_+(1,1)\\
K_+(3,3) &= \frac{\xi(\mathrm{e}^{-h}\sin u - \mathrm{e}^{h}2\zeta_+\cos u)}{\xi'(\mathrm{e}^{h}\sin u + \mathrm{e}^{-h}2\zeta_+\cos u)}K_+(1,1)\\
K_+(4,4) &= \frac{\xi(\mathrm{e}^{h}\sin u - \mathrm{e}^{-h}2\zeta_+\cos u)}{\xi'(\mathrm{e}^{-h}\sin u + \mathrm{e}^{h}2\zeta_+\cos u)}K_+(2,2).
\end{aligned}
\tag{25}
$$

Case II: If $q_+ = -q_- = \zeta_+$

$$
\begin{aligned}
K_+(2,2) &= \frac{\xi(\mathrm{e}^{-h}\sin u + \mathrm{e}^{h}2\zeta_+\cos u)}{\xi'(\mathrm{e}^{h}\sin u - \mathrm{e}^{-h}2\zeta_+\cos u)}K_+(1,1)\\
K_+(3,3) &= \frac{\xi(\mathrm{e}^{-h}\sin u - \mathrm{e}^{h}2\zeta_+\cos u)}{\xi'(\mathrm{e}^{h}\sin u + \mathrm{e}^{-h}2\zeta_+\cos u)}K_+(1,1)\\
K_+(4,4) &= \frac{\xi(\mathrm{e}^{h}\sin u - \mathrm{e}^{-h}2\zeta_+\cos u)}{\xi'(\mathrm{e}^{-h}\sin u + \mathrm{e}^{h}2\zeta_+\cos u)}K_+(2,2).
\end{aligned}
\tag{26}
$$

Thus $K_+(u)$ can be expressed as follows.

Case I

$$
\begin{aligned}
K_+(1,1) &= \lambda_+ \cos^2(u-\beta)(\mathrm{e}^{-h}\sin u + \mathrm{e}^{h}2\zeta_+\cos u)(\mathrm{e}^{h}\sin u + \mathrm{e}^{-h}2\zeta_+\cos u)\\
K_+(4,4) &= \lambda_+ \cos^2(u+\beta)(\mathrm{e}^{h}\sin u - \mathrm{e}^{-h}2\zeta_+\cos u)(\mathrm{e}^{-h}\sin u - \mathrm{e}^{h}2\zeta_+\cos u)\\
K_+(2,2) &= \lambda_+ \cos(u-\beta)\cos(u+\beta)(\mathrm{e}^{-h}\sin u + \mathrm{e}^{h}2\zeta_+\cos u)(\mathrm{e}^{-h}\sin u - \mathrm{e}^{h}2\zeta_+\cos u)\\
K_+(3,3) &= \lambda_+ \cos(u-\beta)\cos(u+\beta)(\mathrm{e}^{-h}\sin u + \mathrm{e}^{h}2\zeta_+\cos u)(\mathrm{e}^{-h}\sin u - \mathrm{e}^{h}2\zeta_+\cos u).
\end{aligned}
\tag{27}
$$

Case II

$$K_+(1,1) = \lambda_+ \cos^2(u-\beta)(\mathrm{e}^h \sin u + \mathrm{e}^{-h} 2\zeta_+ \cos u)(\mathrm{e}^h \sin u - \mathrm{e}^{-h} 2\zeta_+ \cos u)$$
$$K_+(4,4) = \lambda_+ \cos^2(u+\beta)(\mathrm{e}^h \sin u + \mathrm{e}^{-h} 2\zeta_+ \cos u)(\mathrm{e}^h \sin u - \mathrm{e}^{-h} 2\zeta_+ \cos u)$$
$$K_+(2,2) = \lambda_+ \cos(u+\beta)\cos(u-\beta)(\mathrm{e}^{-h} \sin u + \mathrm{e}^{h} 2\zeta_+ \cos u)(\mathrm{e}^h \sin u + \mathrm{e}^{-h} 2\zeta_+ \cos u)$$
$$K_+(3,3) = \lambda_+ \cos(u+\beta)\cos(u-\beta)(\mathrm{e}^{-h} \sin u - \mathrm{e}^{h} 2\zeta_+ \cos u)(\mathrm{e}^h \sin u - \mathrm{e}^{-h} \zeta_+ \cos u). \tag{28}$$

Where λ_+ and ζ_+ are also arbitrary constants describing the boundary effect. Here we note that $K_+(u)$ cannot be obtained from an isomorphism of matrices $K_+(u)$ and $K_-(u)$. But the $K_\pm(u)$ matrices indeed describe the independent BC on each end compatible with the integrability of the model and provide the different boundary terms in the Hamiltonian (7) and (9). We can show that the Hamiltonian (9) is related to the transfer matrix $\tau(u)$ (3) in the following way

$$\tau(u) = C_1 u + C_2 u^2 + C_3(H + \text{constant})u^3 + \cdots$$

here C_i $(i = 1, 2, \ldots)$ are some scalar functions of boundary constants $p_\pm$and $q_\pm$. We wish to stress that although we have not succeeded in finding the $R(u)$ [26], which involve the chemical potential μ, to Yang–Baxter relation:

$$R_{12}(u_1, u_2)\ \overset{1}{T}(u_1)\overset{2}{T}(u_2) = \overset{2}{T}(u_2)\overset{1}{T}(u_1)\ R_{12}(u_1, u_2) \tag{29}$$

where

$$\overset{1}{X} \equiv X \otimes \mathrm{i\,d}_{V_2} \qquad \overset{2}{X} \equiv \mathrm{i\,d}_{V_1} \otimes X \qquad X \in \mathrm{End}(V)$$

for a special 2D statistical model generated by the local monodromy matrix (11), we have shown that the model with both periodic and open BC are completely integrable. We may conjecture that the R-matrix does not possess the crossing unitary.

5. Conclusion

We have presented the Lax pair formulation for 1D Hubbard spin chain with open BC with chemical potential. The explicit form of the Lax pair and two kinds of boundary K-matrices compatible with the integrability of the model have been obtained. Those seem to be nontrivial to derive the Bethe ansatz equations from either the algebraic or analytical Bethe ansatz approach. It shows that the model is related to a class of commuting transfer matrices of an equivalent coupled asymmetric six-vertex spin chain with open BC which can be considered as the generating function of an infinite number of conserved quantities. Thus, the L-operator (11) provided a natural description for the model. To conclude, we wish to point out that the isomorphism between the associative algebras defined by the reflection equations satisfied by $K_\pm$ is induced by the crossing symmetry enjoyed by the quantum R-matrix as well as the monodromy matrix. However, this symmetry does not always exist. This model provides an explicit description of such a situation

Acknowledgments

The auther X-WG would like to thank Professor Zhou for his helpful discussions. This work was supported in part by the National Science Foundation of China.

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